Nonzero Free-Free Frequencies of Structures Idealized by Matrix Methods

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Background

THE stiffness matrix of a free-free structure idealized by finite element methods is singular and, thus, can not be inverted to obtain its flexibility matrix. However, a number of numerical algorithms require the flexibility matrix for extraction of the lower natural modes.[1,2] Mack,[3] Rodden,[4] and Berman and Sklerov[5] overcame this problem either through formulation in terms of relative coordinates or through a "freeing" transformation of the structure's flexibility matrix associated with its being held fixed in some manner. The present Note develops an alternate method for treating this same problem through direct reduction of the free-free systems's stiffness and mass matrices in such a way as to eliminate its zero frequencies. The reduced stiffness matrix may then be inverted, since it is positive definite, and the inertial modal coordinates may be obtained directly as the eigenvectors of the reduced problem.

Method

The governing matrix equation for free vibration is

$$[k]\{x\} = \omega^2[m]\{x\} \tag{1}$$

where $[k]$ and $[m]$ are the structure's assembled stiffness and mass matrices, respectively; the elements of $\{x\}$ are the nodal displacements or rotations; and ω is any one of the system's natural frequencies.

Equation (1) can be partitioned in such a way as to distinguish between $\{x\}$ vector subsets, 1 and 2, i.e.,

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{2a}$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{2b}$$

$$[m] = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \tag{2c}$$

The number of nodal degrees of freedom r comprising $\{x_2\}$ corresponds to the number of zero system frequencies and corresponding rigid-body modes (e.g., for an internally stable three-dimensional structure r would equal 6).

In general, it is assumed that the complete system matrix $[k]$ is only positive, but the submatrices $[k_{11}]$, $[m_{11}]$, and $[m_{22}]$ are positive definite,[6] i.e., the structure is statically determinate and stable if held such that $\{x_2\} = 0$.

Let the structure be subjected to an arbitrary rigid-body motion, $\{x\} = \{x\}_0$. The resulting nodal motions in the $\{x_1\}$ degrees of freedom, $\{x_1\}_0$, can then be expressed in terms of the $\{x_2\}$ nodal motions through

$$\{x_1\}_0 = [T_{12}]\{x_2\}_0 \tag{3}$$

where $[T_{12}]$ is a rigid-body transformation matrix that is readily derivable from kinematical considerations (also see the Appendix). A general rigid-body mode, $\{x\}_0$, therefore

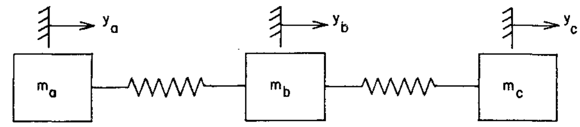

Fig. 1 Three-degree-of-freedom free-free system.

can be represented by

$$\{x\}_0 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_0 = \begin{bmatrix} T_{12} \\ I_{22} \end{bmatrix} \{x_2\}_0 \tag{4}$$

where $[I_{22}]$ is an identity matrix of the same order as $[k_{22}]$.

Substituting Eq. (4) into Eq. (1) yields

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} T_{12} \\ I_{22} \end{bmatrix} \{x_2\}_0 = \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} T_{12} \\ I_{22} \end{bmatrix} \{x_2\}_0 \tag{5}$$

Since a rigid-body mode has been used in Eq. (5), $\omega^2 = 0$, whereas $\{x_2\}_0$ is arbitrary. It therefore follows that

$$[k] \begin{bmatrix} T_{12} \\ I_{22} \end{bmatrix} = [0] \tag{6}$$

Physically speaking, Eq. (6) implies that the stiffness matrix $[k]$ is singular and satisfies dynamic equilibrium requirements for the system.

Turning to the case of a nonrigid-body mode ($\omega^2 \neq 0$), premultiplication of Eq. (1) by $[T_{12}{}^T \vdots I_{22}]$ yields, by virtue of Eq. (6), equivalent conversation of momentum equations

$$[T_{12}{}^T \vdots I_{22}] \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\},\ \omega^2 \neq 0 \tag{7}$$

from which it follows that

$$\{x_2\} = -[T_{12}{}^T m_{12} + m_{22}]^{-1}[T_{12}{}^T m_{11} + m_{21}]\{x_1\},\ \omega^2 \neq 0 \tag{8}$$

Substituting Eq. (8) into Eq. (1) now permits elimination of the vector $\{x_2\}$ and results in the reduced eigenproblem

$$[K_R]\{x_1\} = \omega^2[M_R]\{x_1\},\ \omega^2 \neq 0 \tag{9}$$

where

$$[K_R] = [k_{11} + k_{12}R] \tag{10a}$$

$$[M_R] = [m_{11} + m_{12}R] \tag{10b}$$

$$[R] = -[T_{12}{}^T m_{12} + m_{22}]^{-1}[T_{12}{}^T m_{11} + m_{21}] \tag{10c}$$

The reduced stiffness matrix $[K_R]$ is, in general, positive definite and so may be inverted. Further, inversion of the first matrix appearing in Eq. (10c) usually will involve very little computational labor since this matrix is of the same order as $[k_{22}]$ (no more than 6×6 for an internally stable structure). If, as is often the case, the mass matrix is diagonal in form, i.e., $[m_{12}] = 0$, then the aforementioned inversion becomes trivial, since it is accomplished by taking the reciprocals of the $[m_{22}]$ mass elements.

Example

Consider the 3×3 free-free system, shown in Fig. 1, with only one rigid-body mode, say $\{x\}^T = (1,1,1)$. The partitioned equation governing free vibration is

$$\begin{bmatrix} k_a & -k_a & 0 \\ -k_a & (k_a + k_b) & -k_b \\ 0 & -k_b & k_b \end{bmatrix} \begin{Bmatrix} y_a \\ y_b \\ y_c \end{Bmatrix} = \omega^2 \begin{bmatrix} m_a & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_c \end{bmatrix} \begin{Bmatrix} y_a \\ y_b \\ y_c \end{Bmatrix} \tag{11}$$

where the partitioning is theoretically consistent with the previous development but otherwise arbitrary, e.g., $\{x_2\}$ may equal any one of the node deflections y_i, $i = a, b,$ or c.

Comparison of Eqs. (11) and (2) reveals that

$$[k_{11}] = \begin{bmatrix} k_a & -k_a \\ -k_a & (k_a + k_b) \end{bmatrix} \tag{12a}$$

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$$[k_{21}] = [k_{12}]^T = [0 \quad -k_b] \tag{12b}$$

$$[k_{22}] = [k_b] \tag{12c}$$

$$[m_{11}] = \begin{bmatrix} m_a & 0 \\ 0 & m_b \end{bmatrix} \tag{12d}$$

$$[m_{21}] = [m_{12}]^T = [0 \quad 0] \tag{12e}$$

$$[m_{22}] = [m_c] \tag{12f}$$

$$\{x_1\} = \begin{Bmatrix} y_a \\ y_b \end{Bmatrix} \tag{12g}$$

$$[x_2] = \{y_c\} \tag{12h}$$

Consistent with Eq. (3), the rigid-body transformation matrix is given by

$$[T_{12}] = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13}$$

Substitution of Eqs. (12) and (13) into Eqs. (10) gives

$$[R] = -m_c^{-1}[m_a \quad m_b] \tag{13a}$$

$$[K_R] = m_c^{-1}\begin{bmatrix} k_a m_c & -k_a m_c \\ k_b m_a - k_a m_c & (k_a + k_b)m_c + k_b m_b \end{bmatrix} \tag{13b}$$

$$[M_R] = \begin{bmatrix} m_a & 0 \\ 0 & m_b \end{bmatrix} \tag{13c}$$

Therefore, the reduced-system determinantal equation for the two nonzero frequencies becomes

$$D_2 = \begin{vmatrix} k_a - \omega^2 m_a & -k_a \\ k_b(m_a/m_c) - k_a & (k_a + k_b) + k_b(m_b/m_c) - \omega^2 m_b \end{vmatrix} = 0 \tag{14}$$

It is easily shown that the characteristic equation for the same problem when attacked in the unreduced conventional manner is given by $\omega^2 D_2 = 0$, which has the rigid-body zero frequency as one additional solution.

Symmetrical Form of the Reduced System Equations

Since many algorithms for extracting eigenvalues require that the stiffness and mass matrices be symmetric, an alternative formulation of Eq. (9), leading to symmetric reduced stiffness and mass matrices, is presented. Combining Eqs. (2a), (8), and (10c) yields

$$\{x\} = [S]\{x_1\} \tag{15}$$

where

$$[S] = \begin{bmatrix} I_{n-r} \\ \hline R \end{bmatrix} \tag{16}$$

and $[I]_{n-r}$ is an identity matrix with the same order as $[k_{11}]$. Substitution of Eqs. (15) into (1) and premultiplication by $[S]^T$ then yields

$$[\bar{K}_R]\{x_1\} = \omega^2[\bar{M}_R]\{x_1\}, \ \omega^2 \neq 0 \tag{17}$$

where

$$[\bar{K}_R] = [S]^T[k][S] \tag{18a}$$

and

$$[\bar{M}_R] = [S]^T[m][S] \tag{18b}$$

Thus, the reduced system may be defined by the symmetric matrix Eq. (17), or the unsymmetric matrix Eq. (9).

Appendix

The rigid-body transformation matrix $[T_{12}]$ can also be expressed in terms of the stiffness matrix partitions by noting that

$$\left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{21} & k_{22} \end{array}\right]\begin{Bmatrix} x_1 \\ \hline x_2 \end{Bmatrix}_0 = \{0\}$$

from which $\{x_1\}_0 = -[k_{11}]^{-1}[k_{12}]\{x_2\}_0$. Comparing this result with Eq. (3) reveals that

$$[T_{12}] = -[k_{11}]^{-1}[k_{12}]$$

From a computational point of view, however, it should be noted that the evaluation of $[T_{12}]$ as above requires inversion of $[k_{11}]$ which, for large, complex structures, is approximately of the same order as the system's assembled stiffness matrix. Therefore, unless $[k_{11}]^{-1}$ is available from previous calculations, $[T_{12}]$ is best generated directly from kinematical considerations.

References

[1] Tong, K. N., *Theory of Mechanical Vibration*, Wiley, New York, 1960, pp. 198–202.

[2] Wilkinson, J. H., *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, England, 1965.

[3] Mack, C. E., "Tensor Analysis of Aircraft Structural Vibration," Paper 104, Jan. 1948, IAS.

[4] Rodden, W. P., "On Vibration and Flutter Analysis with Free-Free Boundary Conditions," *Journal of the Aeronautical Sciences*, Vol. 28, Jan. 1961, pp. 65–66.

[5] Berman, J. H. and Sklerov, J., "Calculation of Natural Modes of Vibration for Free-Free Structures in Three-Dimensional Space," *AIAA Journal*, Vol. 3, No. 1, Jan. 1965, pp. 158–160.

[6] Crandall, S. H., *Engineering Analysis*, McGraw-Hill, New York, 1956.

Virtual Origins of a Free Plane Turbulent Jet

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Nomenclature

- b_u = jet half-width (measured to location where $U/U_m = \frac{1}{2}$)
- C_1 = location of the geometric origin from the slot made dimensionless by d
- C_2 = location of the kinematic origin from the slot made dimensionless by d
- d = slot width
- K_1 = rate of widening of the jet, defined by Eq. (1)
- K_2 = slope of the centerline velocity decay, defined by Eq. (2)
- $\bar{U}_m$ = velocity at the axis of the jet
- $\bar{U}_0$ = velocity at the mouth of the jet (at $x = 0$)
- x = distance along the axis measured from the mouth of the jet

Two different origins of similarity in turbulent freejets may be defined. One is based on the widening of the jet, the other on the decay of the axial velocity. Tests (for

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